PHYSICAL REVIEW E VOLUME 59, NUMBER 3 MARCH 1999

# Distribution of the area enclosed by a two-dimensional random walk in a disordered medium

K. V. Samokhin*
*Cavendish Laboratory, University of Cambridge, Madingley Road, Cambridge CB3 0HE, United Kingdom*
(Received 11 December 1998)

The asymptotic probability distribution for a Brownian particle wandering in a two-dimensional plane with random traps to enclose the algebraic area $A$ by time $t$ is calculated using the instanton technique. [S1063-651X(99)50403-0]

PACS number(s): 02.50.−r, 05.40.Fb, 05.40.Jc

It is well known that the properties of random walks are dramatically changed by the presence of quenched disorder. In particular, if a walker can be irreversibly trapped at some randomly distributed sites, then the asymptotic probability to return to the starting point (or the total probability to survive until time $t$) is given in $d$ dimensions by $p(t) \sim \exp(-ct^{d/(d+2)})$ [1], while the mean square displacement decreases compared to a pure diffusion: $\langle r^2\rangle \sim t^{2/(d+2)}$ [2]. On the other hand, if a walker is advected by a random force, then the total probability is conserved, but the diffusion coefficient and the mean square displacement acquire logarithmic corrections [3]. Further examples can be found, for instance, in Ref. [4]. Much less, however, is known about the influence of the quenched disorder on the *topological* properties of random walks, which became a subject of theoretical investigation very recently [5]. By ''topological'' properties we mean such characteristics as the winding number of a Brownian particle, or the linking number of a closed polymer, or the algebraic area enclosed by the trajectory of a random walker. In this Rapid Communication, we concentrate on the latter case and calculate the asymptotic distribution of the area swept by a planar random walk wandering in the presence of traps.

The probability distribution for a random walk starting at some point $\mathbf{r}'$ at $t=0$ to end at a point $\mathbf{r}$ after time $t$ obeys the diffusion equation

$$\frac{\partial P}{\partial t} = D\nabla^2 P - U(\mathbf{r})P. \qquad (1)$$

Here $D$ is the diffusion coefficient and $U(\mathbf{r}) = U_0 \Sigma_i \delta(\mathbf{r} - \mathbf{R}_i)$ is the random ''potential,'' which represents the trapping probability per unit time ($U_0>0$). The positions $\mathbf{R}_i$ of traps are distributed uniformly in a plane according to the Poisson law with mean density $\rho$. The probability for a random walk of ''length'' $t$ to enclose the algebraic area $A$ can be obtained by averaging the corresponding $\delta$-function constraint over the solutions of Eq. (1) in a given distribution of traps:

$$\mathcal{P}(A,t|U) \equiv \left\langle \delta\left(A - \frac{1}{2}\int_0^t d\tau \; r^2(\tau)\dot{\theta}(\tau)\right)\right\rangle_{P(\mathbf{r},t;\mathbf{r}',0|U)},$$

where $\theta(t)$ is the angle between the radius-vector $\mathbf{r}(t)$ of the particle and some fixed direction in the plane. Writing the $\delta$ function as an integral over an auxiliary variable $B$, and using the Wiener path-integral formula for the solution of Eq. (1), we arrive at the following expression:

$$\mathcal{P}(A,t|U) = \int_{-\infty}^{\infty} dp\; e^{iBA} \int_{\mathbf{r}(0)=\mathbf{r}'}^{\mathbf{r}(t)=\mathbf{r}} \mathcal{D}\mathbf{r}(\tau)\; e^{-S[\mathbf{r}(\tau)]}, \qquad (2)$$

where

$$S = -\int_0^t d\tau \left(\frac{1}{2D}\dot{\mathbf{r}}^2(\tau) + U(\mathbf{r}(\tau)) + \frac{i}{2}Br^2(\tau)\dot{\theta}(\tau)\right).$$

It is easy to see that the path integral on the right-hand side of Eq. (2) represents the Euclidean Green function $\mathcal{G}_B(\mathbf{r},t;\mathbf{r}',0)$ of a fictitious quantum particle of mass $m = (2D)^{-1}$ moving in the random potential $U(\mathbf{r})$ and in the uniform magnetic field $B$ (we choose the units in which $\hbar = e = c = 1$). Let us now assume that the trajectory is closed ($\mathbf{r}=\mathbf{r}'$) and average the enclosed area distribution over all positions of the starting point: $\langle(\ldots)\rangle_{\mathbf{r}} = (1/\Omega)\int d^2r\,(\ldots)$ ($\Omega$ is the system volume). We have

$$\mathcal{P}(A,t|U) = \int_{-\infty}^{\infty} dB\; e^{iBA} Z(t,B|U),$$

where $Z(t,B)$ is the partition function at inverse ''temperature'' $1/T = t$. Averaging now over the positions of traps, we finally obtain

$$\mathcal{P}(A,t) = \int_{-\infty}^{\infty} dB \int_0^{\infty} dE\; e^{iBA} e^{-Et} N(E,B), \qquad (3)$$

where $N(E,B)$ is the average density of states of a quantum particle described by the Hamiltonian

$$H = D(-i\nabla - \mathbf{A}(\mathbf{r}))^2 + U(\mathbf{r}). \qquad (4)$$

We choose the cylindrical gauge for the vector potential: $A_\theta = Br/2$.

In an ideal case (i.e., in the absence of traps), the eigenvalues of the Hamiltonian (4) are the Landau levels $E_n = \omega_c(n+1/2)$, where $\omega_c = 2DB$ is the cyclotron frequency. The density of states is then given by the set of equidistant

*Permanent address: L. D. Landau Institute for Theoretical Physics, Kosygina Str. 2, 117940 Moscow, Russia.

 

$\delta$-function peaks, and the partition function can be easily evaluated, resulting in the following expression for the enclosed area distribution:

$$\mathcal{P}(A,t)\sim\frac{1}{\cosh^2 x},\quad x\sim\frac{A}{Dt}. \tag{5}$$

This result was first obtained by Levy quite a while ago [6]. The form of the dimensionless scaling variable $x$ is quite natural, since the only characteristic scale with dimensionality of area in a clean system is the mean square displacement $\langle r^2\rangle\sim Dt$.

In order to calculate the density of states in the presence of disorder, let us first estimate what characteristic scales of energy and magnetic field determine the asymptotic behavior of $\mathcal{P}(A,t)$. We are interested in the calculation of the probability distribution at large but fixed $t$ and $A\to\infty$. As seen from Eq. (3), this limit corresponds to $E\to 0, B\to 0$, and $\omega_c \ll E$. Therefore, it looks natural to start with the case of $B=0$ and make sure that we are able to treat the magnetic field as a small perturbation in the relevant range of parameters.

At $B=0$, the low-energy behavior of the density of states is determined by the rare fluctuations of the concentration of impurities creating the large areas free of traps that are able to sustain the eigenstates with $E\to 0$. The asymptotic expression is given by $N(E,B=0)\sim\exp(-\mathrm{const}\,\rho D/E)$ at $E\ll E_0$, where $E_0=\rho U_0$ is the mean value of the random potential [7]. Quantitatively, such exponentially small "tails" are determined by the contributions of instantons, i.e., spatially localized solutions of the saddle-point equations of the effective field theory. The typical size of a clean area, or the instanton diameter, grows in the time representation as $l_{\mathrm{inst}}\sim(Dt/\rho)^{1/4}$ [2]. If an external magnetic field is imposed on the instanton, then its effect on the energy spectrum can be calculated perturbatively as long as the magnetic length $l_B=\sqrt{1/B}$ considerably exceeds the instanton dimension $l_{\mathrm{inst}}$, which is the case if $A\gg(Dt/\rho)^{1/2}$. It is this condition that determines the limits of applicability of our theory. Note also that if we were interested in the calculation of the intermediate asymptotics of $\mathcal{P}(A,t)$ at moderate $A$, then we could use the exact expressions for the density of states at $E-\omega_c/2\ll\omega_c$ [8] (since the random potential is positive everywhere, the density of states vanishes at $E<\omega_c/2$).

Let us now calculate the effect of an external magnetic field on instantons explicitly. The density of states can be expressed in terms of the retarded Green function of the Schrödinger equation with the Hamiltonian (4): $N(E,B)=-(\pi\Omega)^{-1}\int d^Dr\,\mathrm{Im}\langle G^R(\mathbf{r},\mathbf{r};E,B)\rangle_U$, where $G^R(\mathbf{r},\mathbf{r}';E,B)=\langle\mathbf{r}|(E-H+i0)^{-1}|\mathbf{r}'\rangle$. The Green function can, in turn, be calculated by standard means of the quantum field theory. In order to carry out the averaging over the positions of traps, we resort to the supersymmetry approach, in which the cancellation of denominators is achieved by doubling the degrees of freedom and introducing the commuting and anticommuting fields on equal footing (see, for instance, Ref. [9]). Before disorder averaging, the retarded Green function can be written as the following functional integral:

$$G^R(\mathbf{r},\mathbf{r}';E,B)=-i\lim_{\eta\to+0}\int\mathcal{D}^2\Phi(\mathbf{r})\varphi(\mathbf{r})\overline{\varphi}(\mathbf{r}')\times\exp\left(i\int d^2r\,\overline{\Phi}(E-H+i\eta)\Phi\right), \tag{6}$$

where the two-component superfields

$$\Phi(\mathbf{r})=\begin{pmatrix}\varphi(\mathbf{r})\\ \psi(\mathbf{r})\end{pmatrix},\quad \overline{\Phi}(\mathbf{r})=(\varphi^*(\mathbf{r}),\overline{\psi}(\mathbf{r}))$$

are composed of a Bose field $\varphi$ and a Grassmanian field $\psi$, and $\mathcal{D}^2\Phi=(1/\pi)\mathcal{D}(\mathrm{Re}\,\varphi)\mathcal{D}(\mathrm{Im}\,\varphi)\mathcal{D}\overline{\psi}\mathcal{D}\psi$. The probability of having $N$ impurities located at the points $\mathbf{R}_1,\ldots,\mathbf{R}_N$ in the area $\Omega$ in the plane is given by the Poisson law:

$$P_N(\mathbf{R}_1,\ldots,\mathbf{R}_N)=\frac{e^{-\rho\Omega}}{N!}(\rho\Omega)^N.$$

Using this expression to average Eq. (6) over the positions of traps, we end up with the following effective action:

$$iS[\Phi]=\int d^Dr\{i\overline{\Phi}(E-D(-i\nabla-\mathbf{A})^2+i\eta)\Phi-\rho(1-e^{-iU_0\overline{\Phi}\Phi})\}. \tag{7}$$

An important property of the action (7) is that it is invariant under the global supersymmetry transformations, mixing the boson and fermion sectors:

$$\Phi\to\tilde{\Phi}=T\Phi, \tag{8}$$

where $T$ is a $2\times 2$ unitary supermatrix [9]. Due to this symmetry, the problem of finding the saddle points of Eq. (7) can be considerably simplified. Indeed, since the saddle-point manifold is invariant under the transformations (8), we are able to seek the instanton solution in the following form:

$$\Phi_{\mathrm{inst}}(\mathbf{r})=\begin{pmatrix}\varphi_{\mathrm{inst}}(\mathbf{r})\\ 0\end{pmatrix}, \tag{9}$$

where $\varphi_{\mathrm{inst}}(\mathbf{r})=\varphi_{\mathrm{inst}}(r)$ is a cylindrically symmetric spatially localized function. If the boson fields are written as $\varphi=\varphi_1+i\varphi_2$, $\varphi^*=\varphi_1-i\varphi_2$, where $\varphi_{1,2}$ are real on the initial functional integration contour, then the imaginary part of the Green function is determined by a nontrivial saddle point of the action in the complex plane of $\varphi_{1,2}$ [10,11], and

$$N(E,B)\sim e^{-S_{\mathrm{inst}}(E,B)}, \tag{10}$$

with the exponential accuracy. The fermion sector can be neglected as long as we are not interested in calculation of the pre-exponential factor in Eq. (10).

In order to make Eq. (7) in the boson sector real, we rotate the integration contour: $\varphi_{1,2}\to e^{-i\pi/4}\varphi_{1,2}$. As a result of this, the exponent on the right-hand side of Eq. (6) changes: $iS[\Phi]\to -S[\Phi]$, where the action $S$ is

$$S=\int d^2r\{\varphi_i(-E+D(-i\nabla-\mathbf{A})^2)\varphi_i+\rho(1-e^{-U_0\varphi_i^2})\} \tag{11}$$

$(i=1,2)$. Introducing the dimensionless variables:

$$r=\xi x, \quad \varphi_{\rm inst}(r)=\sqrt{U_0} f(x),$$

where $\xi=\sqrt{D/E}$, we obtain, from Eq. (11), a nonlinear differential equation for the saddle-point solution $f$:

$$-\frac{1}{x}\frac{d}{dx}\left(x\frac{d}{dx}\right)f+\beta^2x^2f+\alpha^2e^{-f^2}f=f, \tag{12}$$

where $\alpha=\sqrt{E_0/E}\gg 1$ and $\beta=\omega_c/4E\ll 1$. Similar equations without a magnetic field were obtained in Refs. [12,13], using different techniques.

Due to complexity of the equation (12), we are able to find only an approximate solution. To this end, we replace the "potential" $V(f)=\alpha^2e^{-f^2}$ in Eq. (12) by a piecewise constant potential:

$$V_{\rm eff}(f)=\begin{cases}\alpha^2, & \text{at } f<1\\ 0, & \text{at } f>1.\end{cases} \tag{13}$$

Then, Eq. (12) becomes effectively linear and reduces to a couple of the Schrödinger equations, whose solutions $f_{1,2}(x)$ satisfy the following conditions:

$$f_1(x_1)=f_2(x_1)=1, \;\; f_1'(x_1-0)=f_2'(x_1+0), \tag{14}$$

where $x_1(\alpha,\beta)$ is the position of the discontinuity in the effective potential, which is to be determined self-consistently. Going back to the dimensional variables, it is easy to see from Eqs. (11) and (12) that the instanton action is proportional to the area of the effective potential well:

$$S_{\rm inst}=\pi\rho\xi^2x_1^2. \tag{15}$$

Let us start with the case of $B=0$, i.e., $\beta=0$. The solution of the linearized saddle-point equations can be written in the form

$$f(x)=\begin{cases}f_1(x)=C_1J_0(x), & \text{at } 0<x<x_1\\ f_2(x)=C_2K_0(\alpha x), & \text{at } x_1<x,\end{cases} \tag{16}$$

where $J_0(x)$ and $K_0(x)$ are the Bessel functions of real and imaginary arguments, respectively. Substituting this solution in the matching conditions (14), we obtain the equation

$$\frac{J_1(x)}{J_0(x)}=\alpha\frac{K_1(\alpha x)}{K_0(\alpha x)}. \tag{17}$$

In the limit $\alpha\to\infty$, assuming that $x_1\sim 1$ and using the asymptotic expansions of the Bessel functions [14], we obtain $J_0(x_1)=0$, i.e., $x_1=a$, where $a\approx 2.405$ is the first zero of the function $J_0(x)$. After substitution into Eqs. (15) and (10), the Lifshitz result $N(E)\sim\exp(-{\rm const}\,\rho D/E)$ is recovered.

In principle, one can find the exact solutions of the Schrödinger equations inside and outside the potential well at $B\neq 0$ (they are expressed in terms of the confluent hypergeometric functions), match them at the point $x=x_1$, and finally end up with a transcendental equation for $x_1(\alpha,\beta)$, which can be solved at $\beta\to 0$. However, we prefer not to follow this procedure here, because the same results can be obtained using much more physically apparent reasoning in the spirit of the Lifshitz original derivation [7]. First we note that, in the limit $\alpha\to\infty$, the instanton solution satisfies the Schrödinger equation for a particle confined in the potential well with infinitely high walls in a magnetic field. The ground state energy $\epsilon_0$ is equal to unity (in the units of $E$). In the absence of a magnetic field this condition fixes the radius of the well at $x_1(0)=a$. At $\beta\neq 0$, the lowest order perturbative correction to the ground state energy is

$$\delta\epsilon_0=\frac{\beta^2\int_0^a dx\, x^3f^2(x)}{\int_0^a dx\, xf^2(x)}, \tag{18}$$

where $f(x)\sim J_0(x)$ is the unperturbed ground state wave function (16). Calculating the integrals with the Bessel functions, we obtain $\delta\epsilon_0=c\beta^2$, where $c\simeq 1.261$. To keep the ground state energy fixed, this correction should be compensated by the corresponding increase in the radius of the potential well: $x_1^2(\beta)=x_1^2(0)(1+\delta\epsilon_0)$. Substituting this into Eq. (15), we finally obtain

$$S_{\rm inst}(E,B)\simeq\frac{\pi\rho Da^2}{E}\left(1+\frac{cD^2}{4E^2}B^2\right). \tag{19}$$

This expression is valid at $E\to 0, B\to 0, \omega_c/E\to 0$.

The asymptotic probability distribution of the enclosed area can now be obtained from Eqs. (3), (10), and (19) by calculating the integrals by the steepest descent method:

$$\mathcal{P}(A,t)\sim\int_{-\infty}^{\infty}dB\int_0^{\infty}dE\, e^{iBA}e^{-Et}e^{-S_{\rm inst}(E,B)}$$
$$\sim\exp(-\sqrt{\pi\rho Da^2t})\exp\left\{-\frac{a\sqrt{\pi}}{c}\frac{\sqrt{\rho}A^2}{(Dt)^{3/2}}\right\}. \tag{20}$$

The first exponential on the right-hand side is nothing but the asymptotic "tail" of the total probability $p(t)=\int dA\,\mathcal{P}(A,t)$ for a random walker not to be trapped [1]. The second exponential thus represents the conditional probability to enclose the area $A$, provided that a walker has survived until time $t$.

We see that the asymptotic behavior of $\mathcal{P}(A,t)$ is drastically changed by the presence of disorder, compared to the Levy result (5). The distribution becomes Gaussian, $\mathcal{P}(A,t)\sim\exp(-x^2)$, with the scaling variable

$$x\sim\frac{A}{Dt}(\rho Dt)^{1/4}, \tag{21}$$

so that the standard deviation now grows slower than in the clean case: $\langle A^2\rangle^{1/2}\sim t^{3/4}$ (the mean value $\langle A\rangle$ is, of course, zero). Such a different form of the scaling variable can be related to the presence of an extra length scale $r_\rho\sim\sqrt{1/\rho}$ in the system, which depends on the concentration of the traps (nothing depends on the absolute value $U_0$ of the random potential). Note also that a similar Gaussian distribution was obtained in Ref. [15] for a two-dimensional random walk in a box of a finite size $L$. In this case, the fictitious magnetic field was also treated perturbatively, giving rise to $\langle A^2 \rangle^{1/2} \sim L(Dt)^{1/2}$. Qualitatively, such a difference between the two systems is due to the fact that in our case the size of the effective potential well is not constant, but grows with time as $L(t) = l_{\mathrm{inst}}(t) \sim t^{1/4}$.

In conclusion, we studied the asymptotic probability distribution of the algebraic area enclosed by a planar random walk in the presence of immobile random traps. It is shown that this probability is directly related to the ''Lifshitz tail'' in the density of states of a quantum particle in a Poisson disorder and uniform magnetic field. Unlike the case of an ideal random walk, the enclosed area distribution turns out to be Gaussian with the standard deviation growing as $t^{3/4}$.

This work was financially supported by the Engineering and Physical Sciences Research Council (U.K.).

[1] B.Ya. Balagurov and V.G. Vaks, Zh. Éksp. Teor. Fiz. **65**, 1939 (1973) [Sov. Phys. JETP **38**, 968 (1974)].
[2] P. Grassberger and I. Procaccia, J. Chem. Phys. **77**, 6281 (1982).
[3] D.S. Fisher, Phys. Rev. A **30**, 960 (1984).
[4] J.P. Bouchaud and A. Georges, Phys. Rep. **195**, 127 (1990).
[5] K.V. Samokhin, J. Phys. A **31**, 8789 (1998); **31**, 9455 (1998).
[6] P. Levy, *Processus Stochastique et Mouvement Brownien* (Gauthier-Villars, Paris, 1948).
[7] I.M. Lifshitz, Usp. Fiz. Nauk **83**, 617 (1964) [Sov. Phys. Usp. **7**, 549 (1965)].
[8] E. Brezin, D. Gross, and C. Itzykson, Nucl. Phys. B **235 [FS11]**, 24 (1984).
[9] K.B. Efetov, *Supersymmetry in Disorder and Chaos* (Cambridge University Press, Cambridge, England, 1997).
[10] J.S. Langer, Ann. Phys. (N.Y.) **41**, 108 (1967).
[11] C. Callan and S. Coleman, Phys. Rev. D **16**, 1762 (1977).
[12] R. Friedberg and J.M. Luttinger, Phys. Rev. B **12**, 4460 (1975).
[13] T.C. Lubensky, Phys. Rev. A **30**, 2657 (1984).
[14] *Handbook of Mathematical Functions* edited by M. Abramowitz and I.A. Stegun (Dover, New York, 1965).
[15] J. Desbois and A. Comtet, J. Phys. A **25**, 3097 (1992).